ANALYTIC, CROSSING-SYMMETRIC, UNITARITY $\pi\pi$ AMPLITUDES WHICH SATISFY $I=1$ SUM RULES AND THE INEQUALITIES OF MARTIN*

E. P. TRYON **

Columbia University, New York, N.Y., USA

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We exhibit a model for analytic, crossing-symmetric $\pi\pi$ amplitudes which satisfy $I = 1$ sum rules and the inequalities of Martin. The S-waves and P-wave are unitary to excellent approximation up to energies exceeding 1 GeV. The model enables us to convert a small number of experimentally accessible parameters into phase shifts reliable up to 1 GeV. Predictions are made for the S-waves which lend support to certain analyses of $\pi N \rightarrow \pi\pi N$ data. Solutions are displayed which constitute counter-examples to the claims of Le Guillou, Morel and Navelet that the S-waves must possess Adler-like zeroes, and that the ϵ resonance must be broad.

In this letter, we display a model for $\pi\pi$ elastic scattering amplitudes in which the following conditions are satisfied *exactly*: analyticity, crossing symmetry, positivity of absorptive parts of partial waves with definite isospin and validity of the Froissart-Gribov representation for partial waves with $l \geqslant 2$. It follows from the preceding properties that the inequalities of Martin [1] and others are satisfied.

Our model also satisfies exactly the $I = 1$ Regge sum rule [2] for the derivative parameter λ_1 of Chew and Mandelstam [3].

In addition to the preceding exact properties, the S-waves and P-wave are unitary to excellent approximation to energies exceeding 1 GeV and the Regge sum rule [4] for the combination $(2a_0 - 5a_2)$ of S-wave scattering lengths is satisfied to excellent approximation.

Our model enables us to convert a small number of experimentally accessible parameters into phase shifts reliable up to 1 GeV. It also sheds light on the physical content of the inequalities of Martin. Several important conclusions are drawn.

Our model may be described briefly as follows (a thorough exposition is in preparation). Let V^I $(\nu, \cos\theta)$ denote the Veneziano $\pi\pi$ amplitude [5] with isospin I in the s-channel, where $\nu \equiv \frac{1}{4}s-1$ (we use units wherein $m_\pi = \hbar = c = 1$). Let $A^I(\nu, \cos\theta)$ denote the physical $\pi\pi$ amplitude. Then consider the functions

$$\Delta A^I(\nu, \cos\theta) \equiv A^I - V^I. \tag{1}$$

If the resonance spectra of the A^I agree with those of the V^I, then resonance contributions to $\mathrm{Im}[\Delta A^I]$ vanish, in the sense of local averages. Thus resonance contributions to all but the nearest singularities of ΔA^I are *effectively zero.*

In our model, we assume for $l \geqslant 2$ that the $\mathrm{Im}A^{(l)I}$ are given for $\nu > 0$ by the δ-function absorptive parts of the Veneziano resonances (which, through duality, contain the contributions of Reggeized ρ and f_0 exchange):

$$l \geqslant 2, \quad \nu > 0: \mathrm{Im}[\Delta A^{(l)I}] = 0\ . \tag{2}$$

Analyticity, crossing symmetry and the approximation (2) imply representations for the ΔA^I which are unique up to the addition of entire functions (only one of which is left independent by crossing symmetry). Our choice of the independent entire function is a simple and natural one whose virtues will

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** Present address: Department of Physics and Astronomy, Hunter College of CUNY, New York, New York 10021.

become apparent†. For $I = \binom{0}{2}$, we have (3a)

$$\Delta A^I(\nu, \cos\theta) = \binom{-5}{-2}\Delta\lambda + \frac{1}{\pi}\int_0^\infty d\nu' \left\{\left[\frac{2(\nu+2\nu'+2)\Delta f^I(\nu,\nu')}{(\nu+2\nu'+2)^2-\nu^2\cos^2\theta} - \frac{\Delta f^I(\nu_0,\nu')}{\nu'-\nu_0}\right] + \frac{\nu-\nu_0}{(\nu'-\nu_0)(\nu'-\nu)}\,\mathrm{Im}\,[\Delta A^{(0)I}(\nu')]\right\},$$

where $\nu_0 \equiv -\frac{2}{3}$, $\Delta\lambda$ is a subtraction parameter which bears the same relation to the ΔA^I as the parameter

$$\lambda \equiv -\tfrac{1}{5}A^0(\nu_0, 0) = -\tfrac{1}{2}A^2(\nu_0, 0)$$

bears to the A^I and Δf^I is given by

$$\Delta f^I(\nu,\nu') = \sum_{I'=0}^{2} \alpha_{II'} \sum_{l'=0}^{1} (2l'+1)\,\mathrm{Im}[\Delta A^{(l')I'}(\nu')]\; P_{l'}\left(1+2\,\frac{\nu+1}{\nu'}\right),$$

where $\alpha_{II'}$ denotes the crossing matrix [3]. For $I = 1$, we have

$$\Delta A^1(\nu, \cos\theta) = \cos\theta\;\frac{\nu}{\pi}\int_0^\infty d\nu' \left\{\frac{2\Delta f^1(\nu,\nu')}{(\nu+2\nu'+2)^2-\nu^2\cos^2\theta} + \frac{3\,\mathrm{Im}[\Delta A^{(1)1}(\nu')]}{\nu'(\nu'-\nu)}\right\}. \tag{3b}$$

The amplitudes (3a-b) are manifestly analytic and free of ghosts (singularities at complex points on the physical sheet). The crossing symmetry is perhaps not obvious, but has been explicitly verified by the present author for arbitrary $\mathrm{Im}[\Delta A^{(0)I}]$, $\mathrm{Im}[\Delta A^{(1)1}]$, $\nu \geq 0$.

It is evident from inspection of eqs. (3a-b) that as $t \to \infty$ for fixed s, the ΔA^I do not grow more rapidly than t. Thus the partial waves $\Delta A^{(l)I}$ satisfy the Froissart-Gribov representation for $l \geq 2$.

Since the Veneziano partial waves $V^{(l)I}$ also satisfy the Froissart-Gribov representation for $l \geq 2$, the $A^I = V^I + \Delta A^I$ generated by our model will automatically satisfy the inequalities of Martin and others if the positivity condition is satisfied: $\mathrm{Im}A^{(l)I}(\nu) \geq 0$ for $\nu \geq 0$ for all l, I.

We shall use the single-term Veneziano formula and the Lovelace values [5] for the trajectory parameters: $a = 0.483$, $b = 0.017$, where $\alpha(s) = a + bs$. We shall constrain the $\mathrm{Im}[\Delta A^{(0)I}]$ and $\mathrm{Im}[\Delta A^{(1)1}]$ in such a way that the positivity condition will be satisfied by all the $\mathrm{Im}A^{(l)I}$ if all the resonance poles in the $V^{(l)I}$ have residues of correct sign above the second (f_0) tower. This is true at least up to the fiftieth tower ‡ (which occurs at 7.5 GeV) and appears to be true for all higher towers [6].

From a Regge analysis of the combination of $\pi\pi$ amplitudes with $I = 1$ in the t-channel, it follows that the traditional derivative parameter [3] λ_1 obeys a sum rule [2], as does the combination of S-wave scattering lengths $(2a_0 - 5a_2)$ [4]. It is straightforward to show that the A^I generated by our model satisfy the sum rule for λ_1 if

$$\int_0^\infty \frac{d\nu}{(\nu-\nu_0)^2}\,\mathrm{Im}[\Delta A^{(1)1}] = 0. \tag{4}$$

Above 1 GeV, we shall constrain $\mathrm{Im}[\Delta A^{(1)1}]$ to tend smoothly to zero. Below 1 GeV, we shall constrain $A^{(1)1}$ to be unitary and contain a ρ resonance. We shall choose the overall multiplicative constant in the V^I to be such that eq. (4) and hence the sum rule [2] for λ_1 are satisfied exactly. We then find that the sum rule [4] for $(2a_0 - 5a_2)$ is satisfied within 0.3%.

By projecting the $l = 0$ and 1 waves out of eqs. (3a-b), respectively, we obtain

$$A^{(0)I}(\nu) = \tag{5a}$$

$$\binom{-5}{-2}\lambda + V^{(0)I}(\nu) - V^I(\nu_0, 0) + \frac{1}{\pi}\int_0^\infty d\nu' \left\{\frac{2}{\nu}Q_0\left(1+2\frac{\nu'+1}{\nu}\right)\Delta f^I(\nu,\nu') - \frac{\Delta f^I(\nu_0,\nu')}{\nu'-\nu_0} + \frac{\nu-\nu_0}{(\nu'-\nu_0)(\nu'-\nu)}\,\mathrm{Im}[\Delta A^{(0)I}(\nu')]\right\},$$

† If one requires that the Froissart-Gribov representation be satisfied for $l \geq 2$, then the right sides of eqs. (3a-b) could only be modified by adding functions which are linear in s, t and u. If one also requires that the Regge sum rule for the derivative parameter λ_1 be satisfied, then eqs. (3a-b) are unique, provided that the overall multiplicative constant in the V^I is such that the Veneziano ρ resonance has the same mass and effective width as the unitarized ρ in $A^{(1)1}$.

‡ Verified by the present author.

Table 1
Low-energy parameters corresponding to the $\pi\pi$ amplitudes whose S-waves are displayed in figs. 1 and 2. The first entries in each column correspond to the solutions a), while the second entries (in parentheses) correspond to the solutions b).

λ	a_0	a_2	λ_1
-0.09	1.07 (1.11)	0.083 (0.081)	0.168 (0.173)
-0.05	0.60 (0.62)	0.018 (0.016)	0.137 (0.141)
-0.01	0.25 (0.26)	-0.046 (-0.047)	0.111 (0.113)
0.03	-0.02 (-0.02)	-0.109 (-0.110)	0.089 (0.090)

$$A^{(1)1}(\nu) = V^{(1)1}(\nu) + \frac{1}{\pi}\int_0^\infty d\nu' \left\{\frac{2}{\nu}Q_1\left(1+2\frac{\nu'+1}{\nu}\right)\Delta f^1(\nu,\nu') + \frac{\nu}{\nu'(\nu'-\nu)}\mathrm{Im}\left[\Delta A^{(1)1}(\nu')\right]\right\} . \tag{5b}$$

The remaining condition to be imposed on our S-waves and P-wave is unitarity:

$$\mathrm{Re}A^{(l)I} = \left\{\mathrm{Im}A^{(l)I}\left[R_l^I(1+1/\nu)^{1/2} - \mathrm{Im}A^{(l)I}\right]\right\}^{1/2} \tag{6}$$

for $\nu > 0$, where R_l^I is the ratio of elastic to total partial-wave cross sections.

To obtain approximate solutions for the $A^{(0)I}$ and $A^{(1)1}$ which simultaneously satisfy eqs. (5a-b) and (6), we represent each $\mathrm{Im}A^{(0)I}$ between threshold and 1.25 GeV by a flexible 39-parameter trial function with correct threshold behaviour and $\mathrm{Im}A^{(1)1}$ between threshold and 1.05 GeV by a flexible 29-parameter trial function with correct threshold behaviour. Above the aforementioned energies, we let the $\mathrm{Im}A^{(l)I}(l = 0, 1)$ tend smoothly to $\mathrm{Im}\,V^{(l)I}$. We then determine the trial-function parameters by requiring that eqs. (5a-b) and (6) be simultaneously satisfied within 2% over a set of closely-spaced mesh points which span the regions where the trial functions are flexible.

We find that solutions exist as λ, m_ϵ, Γ_ϵ, m_ρ and Γ_ρ are independently varied* over substantial ranges of values. Thus our model does not have very restrictive bootstrap properties. However, once the aforementioned parameters have been specified *, the solutions are unique.

For the solutions exhibited in this paper, we hold the ρ parameters fixed at m_ρ = 762 MeV, Γ_ρ = 120 MeV and we set R_l^I equal to unity.

Motivated by analyses of $\pi N \to \pi\pi N$ data [7], we display two types of ϵ resonance in fig. 1. In 1a, δ_0^0 is constrained to equal 75° at 800 MeV and 90° at 960 MeV. In 1b, δ_0^0 is constrained to equal 90° at 730 MeV and 135° at 875 MeV.

We remark that no ϵ(730) solutions exist wherein δ_0^0 pauses or vacillates in its rise from 90° to 180°. In particular, the published ϵ (730) branches [7] of phase-shift analyses of $\pi N \to \pi\pi N$ data are inconsistent with eqs. (5a) and (6). This conclusion would not be affected by moderate ($\leqslant$ 20%) inelasticity below 1 GeV, nor by any amount of inelasticity above 1 GeV. Thus the ϵ (900) branch [7] must be Nature's choice, unless the published error bars along the ϵ (730) branch are several times smaller than the actual errors [8].

In figs. 2a-b, we display the I = 2 S-wave phase shifts δ_0^2 which correspond to the δ_0^0 of figs. 1a-b, respectively.

In table 1, we display the values of the S-wave scattering lengths a_I and the derivative parameter λ_1 which correspond to the solutions displayed in figs. 1 and 2. The functional relation between a_0 and a_2 constitutes the "universal curve" derived earlier by Morgan and Shaw [9].

For the two types of ϵ resonance displayed in fig. 1, we find that $A^{(0)0}$ vanishes on the interval $1.1 \leqslant s \leqslant 1.7$ only if $-0.0026 \leqslant \lambda \leqslant 0.0040$. Thus all solutions with $\lambda > 0.004$ or $\lambda < -0.003$ constitute counterexamples to the claim of Le Guillou, Morel and Navelet [10] (henceforth G.M.N.) that $A^{(0)0}$ must vanish on the interval $1.1 \leqslant s \leqslant 1.7$.

* We find that it is possible to constrain a resonant amplitude at two (but only two) points inside the resonance peak. Thus for example we can constrain $A^{(1)1}$ to have the values appropriate to δ_1^1 = 90° and 135° at $E_{c.m.}$ = = m_ρ and $(m_\rho + \frac{1}{2}\Gamma)$, respectively, for a range of values of the parameters m_ρ and Γ.

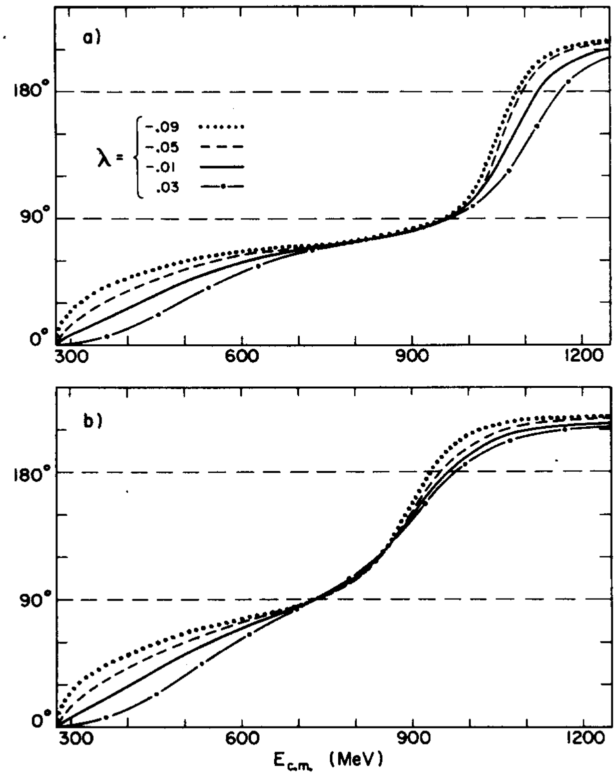

Fig. 1. a) Solutions for δ_0^0 constrained to equal 75° at 800 MeV and 90° at 960 MeV. b) δ_0^0 constrained to equal 90° at 730 MeV and 135° at 875 MeV.

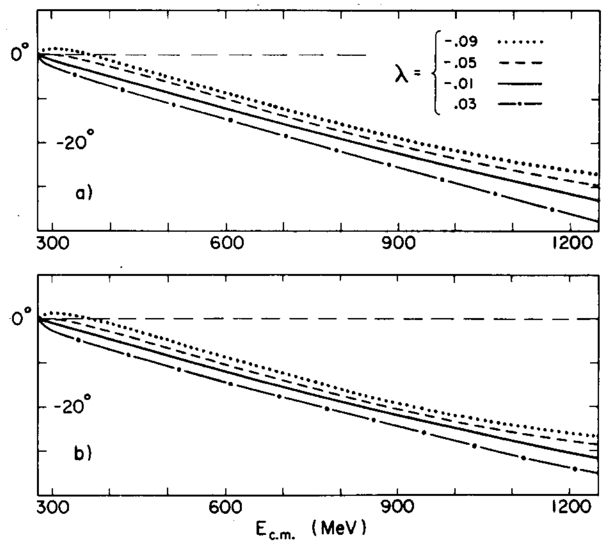

Fig. 2. a) Solutions for δ_0^2 corresponding to the δ_0^0 of fig. 1a. b) δ_0^2 corresponding to the δ_0^0 of fig. 1b.

The δ_0^0 of fig. 1b constitute counterexamples to the argument of G.M.N. against a narrow ϵ ‡. Note, however, that these δ_0^0 all rise through 180°. The S-wave parameterization assumed by G.M.N. does not permit δ_0^0 to reach 180° and this may explain why G.M.N. were unable to obtain a narrow ϵ. A similar remark applies to the work of Carrotte and Johnson [11] and of Gore [12].

Observe that for each value of λ, the δ_0^0 in fig. 1a differs from that in 1b by less than 10% below 400 MeV, while the δ_0^2 in fig. 2a differs from that in 2b by less than 3% below 1 GeV. Thus if one accepts the Weinberg prediction [17] $\lambda \approx -0.008$, which has been confirmed within *statistical* uncertainties of ± 0.01 by analyses of $\pi N \to \pi\pi N$ data [18]†, our model then implies definite values for δ_0^0 and δ_0^2 over the aforementioned ranges of energy. Since the δ_0^2 in fig. 2 do not vary by more than $\pm 3°$ between threshold and 1 GeV as λ is varied over the wide range (-0.01 ∓ 0.04), our model is especially decisive in its prediction of δ_0^2, which we believe to be reliable up to 1 GeV or more [15]. Certain analyses of $\pi N \to \pi\pi N$ data are in good agreement with this prediction and gain support from it ††.

We remark and wish to emphasize that if one had never heard of Veneziano amplitudes but simply wrote twice-subtracted dispersion relations for the $A^{(l)I}$ with subtraction parameters corresponding to λ and λ_1 and then determined λ_1 from the usual sum rule [2], the resulting $A^{(l)I}$ would agree with the amplitudes of our present model within about 10% below 700 MeV for $(l)I = (0)2$ and within about 10%

‡ In the present context, a "narrow" ϵ is one wherein δ_0^0 rises without hesitation toward 180° after reaching 90° near 730 MeV, as opposed to a δ_0^0 which hovers near 90° from about 700 MeV up to at least 900 MeV. The total width of such a "narrow" ϵ may be as much as 400 MeV.

† These authors give values and uncertainties for the a_I which correspond to $\lambda \approx -0.01 \pm 0.01$ in our model. The published uncertainties in the a_I are primarily statistical in origin and do not adequately reflect the theoretical uncertainties in the analyses.

†† The δ_0^2 of Walker et al. [16], Morse et al. [17] and Colton et al. [18] and the recent δ_0^2 of Baton et al. [19] are all in good agreement with our result up to 1 GeV or more. The earlier δ_0^2 of Baton et al. [20] was too small in the ρ region by a factor of two.

below 900 MeV for $(l)I = (0)0$ and $(1)1$ [19]. Thus the amplitudes of our present model have a much greater generality than does the Veneziano model itself.

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